CHAPTER-3

HEAT EQUATIONS

Structure

3.1 Heat Equation – Fundamental solution

3.2 Mean value formula

3.3 Properties of solutions

3.4 Energy methods for Heat Equation

3.1 Definition: The non- homogeneous **Heat (Diffusion) equation** is

$$u_t - \Delta u = f\left(x,t\right) \qquad \ldots (1)$$

where $x \in U \subset R^n$, $f : U \times \left[0,\infty\right) \to R$, $u : \bar{U} \times \left[0,\infty\right) \to R$, the Laplacian Δ is taken w.r.t. spatial variable x, and the function f is given while we have to solve this equation for the unknown function u.

If $f(x,t) = 0$, then the equation

$$u_t - \Delta u = 0 \qquad \ldots (2)$$

is known as homogeneous heat equation.

Physical interpretation: In typical applications, the Heat equation represents the evolution in time of the density u of some quantity such as Heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V .

$$\frac{d}{dt}\int_V u dx = -\int_{\partial V} \vec{F}.\hat{v} ds$$

$\vec{F}$ being the flux density. Thus

$$u_t = -div\vec{F} \qquad \ldots (3)$$

where V is arbitrary.

Theorem: (Fundamental Solution)

Find the fundamental solution of homogeneous Heat equation

$$u_t - \Delta u = 0 \qquad in \qquad \bar{U} \times [0,\infty) \qquad ...(1)$$

where $U \subset R^n$ is open.

Proof: It can be seen from the equation (1) that first order derivate involves w.r.t. to t and second order derivate w.r.t. the space variables $x_1, x_2, ..., x_n$. Consequently, if u solves the equation (1), so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in R$.

So, we seek a solution of equation (1) of the form

$$u(x,t) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right) \quad \text{.... (2)}$$

for $x \in R^n$, $t > 0$. Here, α, β are constants to be determined and the function $v : R^n \to R$ must be find.

Put $y = \frac{x}{t^{\beta}}$ in equation (2), we have

$$u(x,t) = \frac{1}{t^{\alpha}} v(y) \quad \text{... (3)}$$

Differentiating (3) w. r. t. t and x

$$u_t = \frac{-\alpha}{t^{\alpha+1}} v(y) - \frac{\beta y Dv}{t^{\alpha+1}}$$

$$\Delta u = \frac{1}{t^{\alpha+2\beta}} \Delta v$$

Using these expression in equation (1) and simplifying

$$\alpha v(y) + \beta y Dv + \frac{1}{t^{2\beta-1}} \Delta v = 0 \quad \text{... (4)}$$

Now, we simplify the equation (4) by putting $\beta = \frac{1}{2}$, so that the transformed equation involves the variable y only and the equation is

$$\alpha v(y) + \beta y.Dv + \Delta v = 0 \quad \text{... (5)}$$

We seek a radial solution of equation (5) as

$$v(y) = w(r) \quad \text{where } r = |y| \quad \text{....(6)}$$

where $w : R \to R$.

From equation (5) and (6), we have

$$v_{y_i} = w'(|y|)\frac{\partial |y|}{\partial y} = w'(|y|)\frac{y_i}{r} \qquad (\because |y| = r)$$

$$\text{and } v_{y_i y_i} = w''(r)\left(\frac{y_i}{r}\right)\frac{\partial r}{\partial y_i} + w'(r)\left\{\frac{1}{r} - \frac{y_i^2}{r^3}\right\}$$

$$\Delta v(y) = w'' + \omega'(r)\left(\frac{n-1}{r}\right)$$

Using value of $\Delta v(y)$ in equation (4), we get

$$w'' + \left(\frac{r}{2} + \frac{n-1}{r}\right)w' + \alpha w = 0 \qquad \ldots \text{ (7)}$$

Now, if we set $\alpha = \frac{n}{2}$ and multiply by r^{n-1} in equation (7).

Then we have

$$\left(r^{n-1}w'\right)' + \frac{\left(r^n w\right)'}{2} = 0 \qquad \ldots \text{ (8)}$$

Integrating equation (8)

$$r^{n-1}w' + \frac{r^n w}{2} = a \text{ , where a is a constant}$$

Assuming $\lim\limits_{r\to\infty} w, w' = 0$, we conclude $a = 0$, so

$$w' = -\frac{1}{2}rw \qquad \ldots \text{ (9)}$$

Integrating again, we have some constant b

$$w = be^{-r^2/4} \qquad \ldots. \text{ (10)}$$

where b is the constant of integration.

Combining (2) and (10) and our choices for α, β, we conclude that

$$\frac{b}{t^{n/2}} e \frac{-|x|^2}{4t} \text{ solves the Heat equation (1)}$$

To find b, we normalize the solution

$$\int_{R^n} u(x,t)\,dx = 1$$

$$\frac{b}{t^{n/2}} \int_{R^n} e^{-|x|^2/4t}\,dx = 1$$

$$\frac{b}{t^{n/2}} \left(2\sqrt{\pi t}\right)^n = 1$$

$$b = \frac{1}{(4\pi)^{n/2}}$$

Here the function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}; (x \in R^n, t > 0) \\ 0 \qquad , \quad (x \in R^n, t \le 0) \end{cases}$$

is called the fundamental solution of the Heat equation.

Remarks: (i) Φ is singular at the point (0,0).

(ii) Sometimes, we write $\Phi(x,t) = \Phi(|x|,t)$ to emphasise that the fundamental solution is radial in the variable r.

Theorem: Solution of Initial value problem

Solve the initial value (Cauchy) problem

$$u_t - \Delta u = 0 \qquad in \qquad R^n \times (0,\infty) \qquad ...(1)$$

$$u = g \qquad on \qquad R^n \times \{t = 0\} \qquad ...(2)$$

associated with the homogeneous Heat equation, where $g \in C(R^n) \cap L^\infty(R^n)$.

Proof: Let $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}; \qquad (x \in R^n, t > 0) \qquad \ldots (3)$

be the fundamental solution of the equation (1). From earlier article, we note that $(x,t) \to \Phi(x,t)$ solves the Heat equation away from the singularity (0,0) and thus so does $(x,t) \to \Phi(x-y,t)$ for each fixed $y \in R^n$. Consequently, the convolution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4t}} g(y)\, dy$$

$$= \int_{R^n} \Phi(x-y) g(y)\, dy \qquad \ldots (4)$$

Here, we will show that

(i) $u \in C^\infty(R^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \qquad (x \in R^n, t > 0)$

(iii) $\lim_{(x,t)\to(x^0,0)} u(x,t) = g(x^0)$ for each point $x^0 \in R^n, t > 0$

Proof: (i) Since the function $\frac{1}{t^{n/2}} e^{\frac{-|x|^2}{4t}}$ is infinitely differentiable with uniform bounded derivative of all order on $R^n \times [\delta, \infty)$ for $\delta > 0$.

So $u \in C^\infty\left(R^n \times (0,\infty)\right)$.

(ii) $u_t = \int_{R^n} \Phi_t(x-y,t)\, g(y)\, dy$

$$\Delta u = \int_{R^n} \Delta\Phi(x-y,t)\, g(y)\, dy$$

$\therefore u_t - \Delta u = 0$ (since $\Phi(x-y)$ is a solution of Heat equation)

(iii) Fix $x^0 \in R^n$. Since g is continuous, given $\varepsilon > 0, \exists\ \delta > 0$ such that $\left|g(y) - g(x^0)\right| < \varepsilon$ whenever $\left|y - x^0\right| < \delta, y \in R^n$.

Then if $\left|x - x^0\right| < \frac{\delta}{2}$

$$\left|u(x,t) - g(x^0)\right| = \left|\int_{R^n} \Phi(x-y,t)\left[g(y) - g(x^0)\right] dy\right|$$

$$\leq \int_{B(x^0,\delta)} \Phi(x-y,t)\left|g(y) - g(x^0)\right| dy$$

$$+ \int_{R^n - B(x^0,\delta)} \Phi(x-y,t)\left|g(y) - g(x^0)\right| dy$$

$$= I + J \qquad \ldots (5)$$

Now $I \leq \varepsilon \int_{R^n} \Phi(x-y,t)\, dy = \varepsilon$

Furthermore, if $\left|x - x^0\right| \leq \frac{\delta}{2}$ and $\left|y - x^0\right| \geq \delta$ then

$$\left|y - x^0\right| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}\left|y - x^0\right|$$

Thus $|y - x| \geq \frac{1}{2}\left|y - x^0\right|$

Consequently

$$J \leq 2\|g\|_{L^\infty} \int_{R^n - B(x^0,\delta)} \Phi(x-y,t)\, dy$$

$$\leq \frac{c}{t^{n/2}} \int_{R^n - B(x^0,\delta)} e^{\frac{-|x-y|^2}{4t}} dy$$

$$\leq \frac{c}{t^{n/2}} \int_{R^n - B(x^0,\delta)} e^{\frac{-|y-x^0|^2}{16t}} dy$$

$$= \frac{c}{t^{n/2}} \int_{\delta}^{\infty} e^{\frac{-r^2}{16t}} r^{n-1} dr \to 0 \quad \text{as} \quad t \to 0^+$$

Hence, if $\left|x - x^0\right| < \frac{\delta}{2}$ and t>0 is small enough, $\left|u(x,t) - g(x^0)\right| < 2\varepsilon$.

The relation implies that

$$\lim_{\substack{(x,y)\to(x^0,0) \\ x\in R^n, t\to 0^+}} u(x,t) = g(x^0)$$

Thus, we have proved that equation *u(x,t)* given by equation (4) is the solution of the initial value problem.

Theorem: Non-homogeneous Heat Equation

Solve the initial value problem

$$\begin{aligned} u_t - \Delta u = f \qquad & in \qquad R^n \times (0,\infty) \\ u = 0 \qquad & on \qquad R^n \times \{t = 0\} \end{aligned}$$

associated with the non-homogeneous Heat equation, where $f \in C_1^2\left(R^n \times [0,\infty)\right)$ and f has compact support.

Proof:

Define u as

$$u(x,t) = \int_0^t \frac{1}{\left[4\pi(t-s)\right]^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s) dyds \qquad \left(x \in R^n, t > 0\right) \qquad \ldots(1)$$

$$= \int_0^t \int_{R^n} \Phi(x-y, t-s) f(y,s) dyds \qquad \ldots (2)$$

where $f \in C_1^2\left(R^n \times [0,\infty)\right)$ and f has compact support.

Then

(i) $u \in C_1^2\left(R^n \times (0,\infty)\right)$

(ii) $u_t(x,t)-\Delta u(x,t)=f(x,t) \qquad (x\in R^n, t>0)$

(iii) $\lim\limits_{(x,t)\to(x^0,0)} u(x,t)=0$ for each point $x^0\in R^n \quad (x\in R^n, t>0)$

Proof: (i) Since Φ has a singularity at (0,0) we cannot differentiate under the integral sign. Substituting the variable $x-y=0, t-s=0$ and again converting to original variable.

$$u_t(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\,f(x-y,t-s)\,dyds$$

Since $f\in C^2(R^n\times[0,\infty))$ and $\Phi(y,s)$ is smooth near $s=t>0$, we compute

$$u_t(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\,f_t(x-y,t-s)\,dyds$$

$$+\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy \qquad \text{(By Leibnitz's rule)}$$

$$\frac{\partial^2 u}{\partial x_i\partial x_j}(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\frac{\partial^2}{\partial x_i\partial x_j}f(x-y,t-s)\,dyds \qquad (i,j=1,\ldots,n)$$

Thus, $u_t, D_x^2u\in C^2(R^n\times(0,\infty))$.

(ii) Now

$$u_t(x,t)-\Delta u(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\left[\left(\frac{\partial}{\partial t}-\Delta_x\right)f(x-y,t-s)\right]dyds+\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy$$

$$=\int_\varepsilon^t\int_{R^n}\Phi(y,s)\left[\left(\frac{-\partial}{\partial s}-\Delta_y\right)f(x-y,t-s)\right]dyds$$

$$+\int_0^\varepsilon\int_{R^n}\Phi(y,s)\left[\left(\frac{-\partial}{\partial s}-\Delta_y\right)f(x-y,t-s)\right]dyds \quad +\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy$$

$$=I_\varepsilon+J_\varepsilon+K \qquad \ldots (3)$$

Now

$$|J_\varepsilon|\le\left(\|f_t\|_{L^\infty}+\|D^2f\|_{L^\infty}\right)\int_0^\varepsilon\int_{R^n}\Phi(y,s)\,dyds<\varepsilon c$$

Also, we have

$$I_\varepsilon=\int_\varepsilon^t\int_{R^n}\left[\left(\frac{\partial}{\partial s}-\Delta_y\right)\Phi(y,s)\right]f(x-y,t-s)dyds \;+\int_{R^n}\Phi(y,\varepsilon)\,f(x-y,t-\varepsilon)\,dy$$

$$-\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy$$

$$=\int_{R^n}\Phi(y,\varepsilon)\,f(x-y,t-\varepsilon)\,dy-K \qquad \ldots (4)$$

Since Φ solves the Heat equation.

Combining (2) –(4), we have

$$u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon \to 0} \int_{R^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon) dy$$

$$= f(x,y) \qquad \left(x \in R^n, t > 0\right)$$

(iii) $$u(x,t) = \int_0^t \int_{R^n} \Phi(y,s) f(x-y,t-s) dyds$$

$$\|u\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)} \int_0^t \int_{R^n} \Phi(y,s) dyds$$

$$= \|f\| \int_0^t ds = \|f\| t$$

Taking limit as $t \to 0$

$$\lim_{t \to 0} u(x,t) = 0 \text{ for each } x \in R^n .$$

3.2 Mean-Value Formula for the Heat Equation

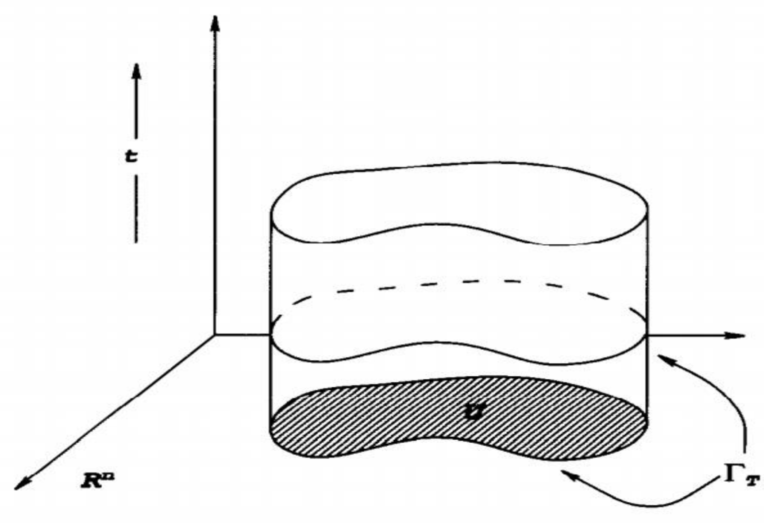
Let $U \subset R^n$ be open and bounded. Fix a time $T > 0$.

Definition: The parabolic cylinder is defined as

$$U_T = U \times (0,T]$$

and the parabolic boundary of U_T is denoted by Γ_T and is defined as

$$\Gamma_T = (\bar{U}_T) - (U_T)$$



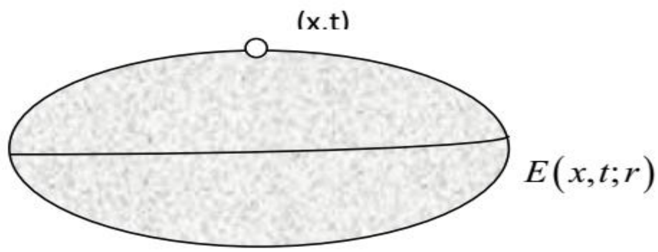
The region U_T

Interpretation: We interpret U_T as being the parabolic interior of $\bar{U}\times[0,T]$. We must note that U_T include to top $U\times\{t=T\}$. The parabolic boundary Γ_T comprises the bottom and vertical sides of $U\times[0,T]$, but not the top.

Definition (Heat ball): For fixed $x\in R^n, t\in R$ and $r>0$, we define

$$E(x,t;r)=\left\{(y,s)\in R^{n+1}\middle| s\le \text{t} \ \text{ and } \Phi(x-y,t-s)\ge\frac{1}{r^n}\right\}$$

$E(x,t;r)$ is a region in space-time. Its boundary is a level set of fundamental solutions $\Phi(x-y,t-s)$ for the Heat equation. The point (x,t) is at the centre of the top. $E(x,t;r)$ is called a Heat ball.



Heat Ball

3.2.1 Theorem: Mean-Value Property for the Heat Equation

Prove that

$$u(x,t)=\frac{1}{4r^n}\iint_{E(x,t:r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \qquad \dots (1)$$

for each Heat ball $E(x,t;r)\subset U_T$. It is assumed that $u\in C_1^2(U_T)$ solve the homogeneous Heat equation

$$u_t-\Delta u=0 \ \text{ in } \ R^n\times(0,\infty) \qquad \dots (2)$$

Proof: The formula (1) is known as mean-value formula. We find that the right hand side of (1) involves only $u(y,s)$ for times $s\le t$. It is reasonable, as the value $u(x,t)$ should not depend upon future times. We may assume upon translating the space and time coordinates that

$$x=0, t=0 \qquad \dots (3)$$

So we can write Heat ball as

$$E(r)=E(0,0;r) \qquad \dots (4)$$

and set

$$\phi(r) = \frac{1}{r} \iint_{E(r)} u(y,s) \frac{|y|}{s^2} dyds$$

$$= \iint_{E(1)} u\left(ry, r^2 s\right) \frac{|y|^2}{s^2} dyds \qquad \text{(by shifting the variable)} \qquad \ldots (5)$$

Differentiating (5), we obtain

$$\phi'(r) = \iint_{E(1)} \left\{ \sum_{i=1}^{n} y_i u_{y_i} \left(\frac{|y|^2}{s^2} \right) + 2ru_s \left(\frac{|y|^2}{s} \right) \right\} dyds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left\{ y_i u_{y_i} \left(\frac{|y|^2}{s^2} \right) + 2u_s \left(\frac{|y|^2}{s} \right) \right\} dyds \quad \text{(Again shifting to original ball)}$$

$$= A + B \qquad \ldots (6)$$

We introduce the useful function

$$\psi = -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r \qquad \ldots (7)$$

Then

$$\psi = 0, \qquad \text{on} \qquad \partial E(r) \qquad \ldots (8)$$

Since,

$$\Phi(y, -s) = r^{-n} \quad \text{on} \quad \partial E(r) \qquad \ldots (9)$$

be definition of Heat ball.

Now, we use (7) to write

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i} dyds$$

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} 4nu_s \psi + 4 \sum_{i=1}^{n} u_{sy_i} y_i \psi dyds \qquad \ldots (10)$$

There is no boundary term since $\psi = 0$ on $\partial E(r)$.

Integrating by parts with respect to s, we find

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i \psi_s dyds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i \left(\frac{-n}{2s} - \frac{|y|^2}{4s^2} \right) dyds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i dyds - A$$

This implies

$$\phi'(r) = A + B$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left\{ -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \right\} dyds$$

$$= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i dyds = 0$$

Therefore, ϕ is constant.

Thus $$\phi(r) = \lim_{t\to 0}\phi(t) = u(0,0)\left\{ \lim_{t\to 0}\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2} dyds \right\} = 4u(0,0) \qquad \ldots(11)$$

$$\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4 \qquad \ldots(12)$$

From equation (4) and (11), we write

$$u(x,t) = \frac{1}{4}\phi(r) \qquad \ldots(13)$$

From (5) and (13), we have

$$u(x,t) = \frac{1}{4r^n}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2} dyds \qquad \ldots(14)$$

Hence proved.

3.3 Properties of Solution

3.3.1 Theorem: Strong Maximum Principle for the Heat Equation

Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the Heat equation in U_T. Then

(i) $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$

(ii) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\bar{U}_T} u$$

Then u is constant in $\bar{U}_{t_0}$.

Proof: Suppose there exists a point $(x_0, t_0) \in U_T$ with

$$u(x_0, t_0) = M = \max_{\bar{U}_T} u$$

It means that the maximum value of u occur at the point (x_0,t_0).

Then for all sufficiently small r>0,

$$E(x_0, t_0; r) \subset U_T$$

By using the mean-value property, we have

$$M = u(x_0, t_0)$$

$$= \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds \le M \qquad \dots (1)$$

Since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds$$

Form equation (1), it is clear that equality holds only if u is identically equal to M within $E(x_0, t_0; r)$. Consequently

$$u(y,s) = M \quad \text{for all } (y,s) \in E(x_0, t_0; r)$$

Draw any line segment L in U_T connecting (x_0, t_0) with some other point $(y_0, s_0) \in U_T$, with $s_0 < t_0$. Consider

$$r_0 = \min\left\{ s \ge s_0 \,\middle|\, u(x,t) = M \;\; for \;\; all \;\; point\, s(x,y) \in L, s \le t \le t_0 \right\}$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$.Then

$$u(z_0, r_0) = M$$

for some point (z_0, r_0) on $L \cap U_T$ and so

$$u \equiv M \;\text{ on }\; E(z_0, r_0; r) \text{ for all sufficiently small r>0}$$

Since $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \le t \le r_0\}$ for some small $\sigma > 0$, which is a contradiction.

Thus

$$r_0 = s_0$$

Hence

$$u \equiv M \;\text{ on }\; L \qquad \dots(2)$$

Now fix any point $x \in U$ and any time $0 \le t \le t_0$. There exists points $\{x_0, x_1, ..., x_m, x\}$ such that the line segments in R^n connecting x_{i-1} to x_i lie in U for $i = 1, ..., m$. (This follows since the set of points in U which can be so connected to x_0 by a polygonal path is nonempty, open and relatively closed in U). Select times $t_0 > t_1 > ... > t_m = t$. Then the line segments in R^{n+1} connecting (x_{i-1}, t_{i-1}) to $(x_i, t_i)(i = 1, ..., m)$ lie in U_T. According to step 1, $u \equiv M$ on each such segment and so $u(x,t) = M$.

Remark: 1. From a physical perspective, the maximum principle states that the temperature at any point x inside the road at any time $(0 \le t \le T)$ is less than the maximum of the initial distribution or the maximum of temperature prescribed at the ends during the time interval [0,t].

2. The strong maximum principle implies that if U is connected and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad U_T \\ u = 0 & \text{on} \quad \partial U \times [0, T] \\ u = g & \text{on} \quad U \times \{t = 0\} \end{cases}$$

where $g \ge 0$, then u is positive everywhere within U_T if g is positive somewhere on U . This is another illustration of infinite propagation speed for disturbances.

3. Similar results holds for minimum principle just by replacing “max” with “min”.

3.3.2 Theorem: Uniqueness on bounded domains

Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem

$$\begin{cases} u_t - \Delta u = f \ \ in\, U_T \\ u = g \quad on\, \Gamma_T \end{cases} \qquad \dots (1)$$

Proof: If $u = \tilde{u}$ are two solutions of (1). Then

$$\begin{cases} u_t - \Delta u = f \ \ in\, U_T \\ u = g \quad on\, \Gamma_T \end{cases} \qquad \dots (2)$$

and

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = f \ \ in\, U_T \\ \tilde{u} = g \quad on\, \Gamma_T \end{cases} \qquad \dots (3)$$

Let $w = \pm(u - \tilde{u})$, then from equation (2) and (3), we have

$$w_t - \Delta w = (u_t - \tilde{u}_t) - \Delta(u - \tilde{u}) = 0$$

$$w = 0 \quad on \quad \Gamma_T$$

apply previous theorem to $w = \pm(u - \tilde{u})$ to get the result.

3.3.3 Regularity

Theorem: Smoothness

Suppose $u \in C_1^2(U_T)$ solves the heat equation in U_T. Then

$$u \in C^\infty(U_T)$$

This regularity assertion is valid even if u attains non-smooth boundary value on Γ_T.

Proof: We write

$$C(x,t;r) = \{(y,s) \| x - y| \le r, t - r^2 \le s \le t\}$$

To denote the closed circular cylinder of radius r, height r^2, and top centre point (x,t). Fix $(x_0, t_0) \in U_T$ and choose $r > 0$ so small that $C = C(x_0, t_0; r) \subset U_T$.

Define also the smaller cylinder

$$C' = C\left(x_0, t_0; \frac{3}{4}r\right), C'' = C\left(x_0, t_0; \frac{r}{2}\right),$$

which have the same top centre point (x_0, t_0). Extend $\zeta \equiv 0$ in $(R^n \times [0, t_0]) - C$

Assume that $u \in C^\infty(U_T)$ and set $v(x,t) = \zeta(x,t)u(x,t) \quad (x \in R^n, 0 \le t \le t_0)$

Then

$$v_t = \zeta u_t + \zeta_t u, \Delta v = \zeta \Delta u + 2D\zeta . Du + u\Delta\zeta$$

Consequently

$$v = 0 \text{ on } R^n \times \{t = 0\} \qquad \dots (1)$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta . Du - u\Delta\zeta = \tilde{f} \quad \text{in} \quad R^n \times (0, t_0)$$

Now, set

$$\tilde{v}(x,t) = \int_0^t \int_{R^n} \Phi(x - y, t - s)\, \tilde{f}(y,s)\, dyds$$

We know that

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} \ in \ R^n \times (0,t_0) \\ \tilde{v} = 0 \quad on \, R^n \times \{t=0\} \end{cases} \qquad \ldots (2)$$

Since $|v|, |\tilde{v}| \le A$ for some constant A, previous theorem implies $v \equiv \tilde{v}$, i.e.

$$v(x,t) = \int_0^t \int_{R^n} \Phi(x-y,t-s)\, \tilde{f}(y,s)\, dyds$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ of the cylinder C, (1) and (3) imply

$$u(x,t) = \iint_C \Phi(x-y,t-s)\left[\left(\zeta_s(y,s) - \Delta\zeta(y,s)\right)u(y,s) - 2D\zeta(y,s).Du(y,s)\right]dyds$$

Integrate the last term by parts:

$$u(x,t) = \iint_C \left[\Phi(x-y,t-s)\left(\zeta_s(y,s) + \Delta\zeta(y,s) + 2D_y\Phi(x-y,t-s).D\zeta(y,s)\right)\right]u(y,s)\,dyds$$

If u satisfies only the hypotheses of the theorem, we derive (4) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

Formula (4) has the form

$$u(x,t) = \iint_C K(x,t,y,s)\,u(y,s)\,dyds \qquad \left((x,t) \in C''\right)$$

where

$$K(x,t,y,s) = 0 \text{ for all points } (y,s) \in C'$$

Since $\zeta = 1$ on C'.

Note that K is smooth on $C - C'$.

We see u is C^∞ within $C'' = C\left(x_0, t_0; \frac{1}{2}r\right)$

Theorem: Local Estimate for Solutions of the Heat Equation

There exists for each pair of integers k, l=0,1,…, a constant $C_{k,l}$ such that

$$\max_{C\left(x,t;\frac{r}{2}\right)} \left|D_x^k D_t^l u\right| \le \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}$$

for all cylinder $C\left(x,t; r/2\right) \subset C(x,t;r) \subset U_T$ and all solutions u of the Heat equation in U_T.

Proof: Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is (0,0).

Suppose first that the cylinder $C(1)=C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right)=C\left(0,0;\frac{1}{2}\right)$

Then

$$u(x,t)=\iint_{C(1)} K(x,t,y,s)u(y,s)\,dyds \qquad \left((x,t)\in C\left(\frac{1}{2}\right)\right)$$

for some smooth function K.

Consequently,

$$\left|D_x^k D_t^l u(x,t)\right| \le \iint_{C(1)} \left|D_t^l D_x^k K(x,t,y,s)\right|\left|u(y,s)\right| dyds$$

$$\le C_{kl}\,\|u\|_{L^1(C(1))}$$

for some constant C_{kl}.

Now suppose the cylinder $C(r)=C(0,0;r)$ lies in U_T. Let $C\left(r/2\right)=C\left(0,0;r/2\right)$.

We define

$$v(x,t)=u\left(rx,r^2t\right)$$

Then $v_t-\Delta v=0$ in the cylinder $C(1)$.

According to (1)

$$\left|D_x^k D_t^l v(x,t)\right| \le C_{kl}\,\|v\|_{L^1(C(1))} \qquad \left((x,t)\in C\left(\frac{1}{2}\right)\right)$$

But $$D_x^k D_t^l v(x,t)=r^{2l+k} D_x^k D_t^l u\left(rx,r^2t\right)$$

and $$\|v\|_{L^1(C(1))}=\frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$$

Therefore,

$$\max_{C\left(r/2\right)}\left|D_x^k D_t^l u\right| \le \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}$$

Note: If u solves the Heat equation within U_T, then for each fixed time $0<t\le T$, the mapping $x\mapsto u(x,t)$ is analytic. However the mapping $t\mapsto u(x,t)$ is not in general analytic.

3.4 Energy Methods

(a) Uniqueness

Theorem: There exists at most one solution $u \in C_1^2\left(\bar{U}_T\right)$ of

$$\begin{cases} u_t - \Delta u = f \;\; in\, U_T \\ \quad u = g \quad\;\; on\, \Gamma_T \end{cases} \qquad \ldots \; (1)$$

Proof: If $\tilde{u}$ be another solution, $w = u - \tilde{u}$ solves

Set

$$e(t) = \int_U w^2(x,t)\,dx \qquad \left(0 \le t \le T\right)$$

Then

$$\begin{aligned} \dot{e}(t) &= 2\int_U w w_t dx \\ &= 2\int_U w \Delta w dx \\ &= -2\int_U |Dw|^2\, dx \le 0 \end{aligned}$$

and so

$$e(t) \le e(0) = 0 \qquad \left(0 \le t \le T\right)$$

Consequently $w = u - \tilde{u}$ in U_T.

(b) Backwards Uniqueness

For this, suppose u and $\tilde{u}$ are both smooth solutions of the Heat equation in U_T, with the same boundary conditions on ∂U .

$$\begin{cases} u_t - \Delta u = 0 \; in \quad\; U_T \\ \quad u = g \quad on\, \partial U \times [0,T] \end{cases} \qquad \ldots (1)$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 \; in \quad\; U_T \\ \quad \tilde{u} = g \quad on\, \partial U \times [0,T] \end{cases} \qquad \ldots \; (2)$$

for some function g.

Theorem: Suppose $u, \tilde{u} \in C^2\left(\bar{U}_T\right)$ solve (1) and (2). If $u(x,t) = \tilde{u}(x,t) \quad \left(x \in U\right)$ then

$$u \equiv \tilde{u} \text{ within } U_T.$$

Proof: Write $w = u - \tilde{u}$ and set

$$e(t) = \int_U w^2(x,t)\,dx \qquad \left(0 \le t \le T\right)$$

Then

$$\dot{e}(t) = -2\int_U |Dw|^2\, dx \qquad \ldots \quad (3)$$

Also

$$\ddot{e}(t) = -4\int_U Dw.Dw_t dx$$

$$= 4\int_U \Delta w w_t dx \qquad \ldots (4)$$

$$= 4\int_U (\Delta w)^2 dx$$

Since w=0 on ∂U ,

$$\int_U |Dw|^2 dx = -\int_U w\Delta w dx$$

$$\le \left(\int_U w^2 dx\right)^{1/2} \left(\int_U (\Delta w)^2 dx\right)^{1/2}$$

From (3) and (4)

$$(\dot{e}(t))^2 = 4\left(\int_U \left|Dw^2\right| dx\right)^2$$

$$\le \left(\int_U w^2 dx\right)\left(4\int_U (\Delta w)^2 dx\right)$$

$$= e(t)\ddot{e}(t)$$

Hence

$$\ddot{e}(t)e(t) \ge (\dot{e}(t))^2 \qquad (0 \le t \le T) \qquad \ldots (5)$$

Now if $e(t) = 0$ for all $0 \le t \le T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$ with

$$e(t) > 0 \text{ for } t_1 \le t \le t_2,\ e(t_2) = 0 \qquad \ldots (6)$$

Write

$$f(t) = \log e(t) \qquad (t_1 \le t \le t_2) \qquad \ldots (7)$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \ge 0$$

If $0 < \tau < 1, t_1 < t < t_2$ then

$$f((1-\tau)t_1 + \tau t) \le (1-\tau) f(t_1) + \tau f(t)$$

Also

$$e((1-\tau)t_1 + \tau t) \le e(t_1)^{1-\tau} e(t)^{\tau},$$

and so

$$0 \le e((1-\tau)t_1 + \tau t_2) \le e(t_1)^{1-\tau} e(t_2)^{\tau} \quad (0 < \tau < 1)$$

This inequality implies $e(t) = 0$ for all times $t_1 \le t \le t_2$, a contradiction.

CHAPTER-4

WAVE EQUATIONS

Structure

4.5 Wave Equation

The homogeneous Wave equation is

$$u_{tt} - \Delta u = 0 \qquad \dots \; (1)$$

and the non-homogeneous Wave equation

$$u_{tt} - \Delta u = f \qquad \dots \; (2)$$

Here $t > 0$ and $x \in U$, where $U \subset R^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to R, u = u(x, t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, ..., x_n)$. In equation (2) the function $f : U \times [0, \infty) \to R$ is given.

Remarks: 1. The Wave equation is a simplified model equation for a vibrating string (*n=1*). For *n=2,* it is membrane and it becomes an elastic solid for *n=3. u(x,t)* represents the displacement in some direction of the point *x* at time $t \geq 0$ for different values of *n*.

2. From physical perspective, it is obvious that we need initial condition on the displacement and velocity at time *t=0.*

Solution of Wave equation by spherical means (for n=1)

Theorem: Derive the solution of the initial value problem for one-dimensional Wave equation

$$u_{tt} - u_{xx} = 0 \text{ in } R \times (0, \infty) \qquad \dots \; (1)$$

$$u = g, u_t = h \text{ on } R \times \{t = 0\} \qquad \dots \; (2)$$

where g, h are given at time t=0..

Proof: The PDE (1) can be factored as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0 \qquad \dots \; (3)$$

Set

$$v(x,t)=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u(x,t) \qquad \dots (4)$$

Then, equation (4) becomes

$$v_t(x,t)+v_x(x,t)=0 \qquad (x\in R, t>0) \qquad \dots (5)$$

Equation (5) becomes the transport equation with constant coefficient (b=1).

Let $\quad v(x,0)=a(x) \qquad \dots (6)$

We know that the fundamental solution of the initial-value problem consisting of transport equation (5) and condition (6) is

$$v(x,t)=a(x-t), x\in R, t\geq 0 \qquad \dots (7)$$

Combining equation (4) and (7), we obtain

$$u_t(x,t)-u_x(x,t)=a(x-t) \text{ in } R\times(0,\infty) \qquad \dots (8)$$

Also

$$u(x,0)=g(x) \text{ in } R \qquad \dots (9)$$

By virtue of initial condition (2), Equations (8) and (9) constitute the non-homogeneous transport problem. Hence its solution is

$$u(x,t)=g(x+t)+\int_0^t a\big(x+(s-t)(-1)-s\big)ds$$

$$=g(x+t)+\frac{1}{2}\int_{x-t}^{x+t} a(y)dy \qquad \dots (10) \qquad (\because x+t-2s=y)$$

The second initial condition in (2) imply

$$a(x)=v(x,0)$$

$$=u_t(x,0)-u_x(0,0)$$

$$=h(x)-g'(x), x\in R \qquad \dots (11)$$

Substituting (11) into (10)

$$u(x,t)=g(x+t)+\frac{1}{2}\int_{x-t}^{x+t}\left[h(y)-g'(y)\right]dy$$

$$=\frac{1}{2}\left[g(x+t)+g(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} h(y)dy \qquad \dots (12)$$

for $x\in R, t\geq 0$.

This is the d' Alembert's formula.

Application of d' Alembert's Formula

Initial/boundary-value problem on the half line $R_+ = \{x > 0\}$.

Example: Consider the problem

$$\begin{cases} u_{tt} - u_{xx} & in \quad R_+ \times (0,\infty) \\ u = g, \quad u_t = h & on \quad R_+ \times \{t = 0\} \\ u = 0 & on \quad \{x = 0\} \times (0,\infty) \end{cases} \qquad ...(1)$$

where g, h are given, with

$$g(0) = 0, h(0) = 0. \qquad \dots (2)$$

Solution: Firstly, we convert the given problems on the half-line into the problem on whole of R We do so by extending the functions u, g, h to all of R by odd reflection method as below we set.

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & for\, x \geq 0, t \geq 0 \\ -u(-x,t) & for\, x \leq 0, t \geq 0 \end{cases} \qquad \dots (3)$$

$$\tilde{g}(x) = \begin{cases} g(x) & for\, x \geq 0 \\ -g(x) & for\, x \leq 0 \end{cases} \qquad \dots (4)$$

$$\tilde{h}(x) = \begin{cases} h(x) & for\, x \geq 0 \\ -h(-x) & for\, x \leq 0 \end{cases} \qquad \dots (5)$$

Now, problem (1) becomes

$$\left.\begin{matrix} \tilde{u}_{tt} = \tilde{u}_{xx} \quad in \;\; R \times (0,\infty) \\ \tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h} \, on \, R \times \{t = 0\} \end{matrix}\right\} \qquad \dots (6)$$

Hence, d' Alembert's formula for one-dimensional problem (6) implies

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy \qquad \dots (7)$$

Recalling the definition of $\tilde{u}, \tilde{g}, \tilde{h}$ in equations (3)-(5), we can transform equation (7) to read for $x \geq 0, t \geq 0$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & if \;\, x \geq t \geq 0 \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & if \;\, 0 \leq x \leq t \end{cases} \qquad \dots (8)$$

Formula (8) is the solution of the given problem on the half-line $R_+ = \{x > 0\}$.

Solution of Wave Equation (for n=3)

Theorem: Derive Kirchhoff's formula for the solution of three-dimensional (n=3) initial-value problem

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad R^3 \times (0,\infty) \qquad \ldots(1)$$

$$u = g \quad \text{on} \quad R^3 \times \{t=0\} \qquad \ldots(2)$$

$$u_t = h \quad \text{on} \quad R^3 \times \{t=0\} \qquad \ldots(3)$$

Solution: Let us assume that $u \in C^2\left(R^3 \times [0,\infty)\right)$ solves the above initial-value problem.

As we know

$$U(x;r,t) = \oint_{\partial B(x,r)} u(y,t)\,ds(y) \qquad \ldots(4)$$

defines the average of $u(.,t)$ over the sphere $\partial B(x,r)$. Similarly,

$$G(x;r) = \oint_{\partial B(x,r)} g(y)\,ds(y) \qquad \ldots(5)$$

$$H(x;r) = \oint_{\partial B(x,r)} h(y)\,ds(y) \qquad \ldots(6)$$

We here after regard U as a function of r and t only for fixed x.

Next, set

$$\tilde{U} = rU\,, \qquad \ldots(7)$$

$$\tilde{G} = rG, \tilde{H} = rH \qquad \ldots(8)$$

We now assert that $\tilde{U}$ solve

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \; in & R_+ \times (0,\infty) \\ \tilde{U} = \tilde{G} \quad on & R_+ \times \{t=0\} \\ \tilde{U}_t = \tilde{H} \quad on & R_+ \times \{t=0\} \\ \tilde{U} = 0 \quad on & \{r=0\} \times (0,\infty) \end{cases} \qquad \ldots(9)$$

We note that the transformation in (7) and (8) convert the three-dimensional Wave equation into the one-dimensional Wave equation.

From equation (7)

$$\tilde{U}_{tt} = rU_{tt}$$

$$= r\left[U_{rr} + \frac{2}{r}U_r\right], \text{ Laplacian for n=3}$$

$$= rU_{rr} + 2U_r$$

$$= \left(U + rU_r\right)_r$$

$$= \left(\tilde{U}_r\right)_r = \tilde{U}_{rr} \qquad \ldots (10)$$

The problem (9) is one the half-line $R_+ = \{r \geq 0\}$.

The d' Alembert's formula for the same, for $0 \leq r \leq t$, is

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy \qquad \ldots (11)$$

From (4), we find

$$u(x,t) = \lim_{r\to 0^+} U(x;r,t) \qquad \ldots (12)$$

Equations (7),(8),(11) and (12) implies that

$$u(x,t) = \lim_{r\to 0^+}\left[\frac{\tilde{U}(x;r,t)}{r}\right]$$

$$= \lim_{r\to 0^+}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r}\tilde{H}(y)\,dy\right]$$

$$= \tilde{G}'(t) + \tilde{H}(t) \qquad \ldots (13)$$

Owing then to (13), we deduce

$$u(x,t) = \frac{\partial}{\partial t}\left\{t \oint_{\partial B(x,t)} g(y)\,ds(y)\right\} + \left\{t \oint_{\partial B(x,t)} h(y)\,ds(y)\right\} \qquad \ldots (14)$$

But

$$\int_{\partial B(x,t)} g(y)\,ds(y) = \int_{\partial B(0,1)} g(x+tz)\,ds(z) \qquad \ldots (15)$$

Hence

$$\frac{\partial}{\partial t}\left\{\oint_{\partial B(x,t)} g(y)\,ds(y)\right\} = \oint_{\partial B(0,1)} \{Dg(x+tz)\}.z\,ds(z)$$

$$= \oint_{\partial B(x,t)} Dg(y).\left(\frac{y-x}{t}\right)ds(y) \qquad \ldots (16)$$

Now equation (14) and (16) conclude

$$u(x,t) = \oint_{\partial B(x,t)} \left[g(y) + \{Dg(y)\}.(y-x) + th(y) \right] ds(y) \tag{17}$$

for $x \in R^3, t > 0$.

The formula (17) is called **KIRCHHOFF'S** formula for the solution of the initial value problem (1)-(3) in 3D.

4.6 Non-Homogeneous Problem

Now we investigate the initial-value problem for the non-homogeneous Wave equation

$$\begin{cases} u_{tt} - \Delta u = f \ \ in \ \ R^n \times (0,\infty) \\ u = 0, u_t = 0 \, on \, R^n \times \{t=0\} \end{cases} \qquad \ldots (1)$$

Motivated by Duhamel's principle, which says that one can think of the inhomogeneous problem as a set of homogeneous problems each starting afresh at a different time slice $t = t_0$. By linearity, one can add up (integrate) the resulting solutions through time t_0 and obtain the solution for the inhomogeneous problem. Assume that $u = u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt}(.,s) - \Delta u(.,s) = 0 \quad in \ \ R^n \times (s,\infty) \\ u(.,s) = 0, u_t(.,s) = f(.,s) \, on \, R^n \times \{t=s\} \end{cases} \qquad \ldots (2)$$

and set

$$u(x,t) = \int_0^t u(x,t;s)\,ds \qquad \left(x \in R^n, t \geq 0\right) \qquad \ldots(3)$$

Duhamel's principle asserts that this is solution of equation (1).

Theorem: Solution of Non-homogeneous Wave Equation

Let us consider the non-homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f \ \ in \ \ R^n \times (0,\infty) \\ u = 0, u_t = 0 \, on \, R^n \times \{t=0\} \end{cases} \qquad \ldots (1)$$

$f \in C^{\left[\frac{n}{2}\right]+1}\left(R^n \times [0,\infty)\right)$ and $n \geq 2$. Define u as

$$u(x,t) = \int_0^t u(x,t;s)\,ds \qquad \left(x \in R^n, t \geq 0\right) \qquad \ldots (2)$$

Then

(i) $u \in C^2\left(R^n \times [0,\infty)\right)$

(ii) $u_{tt} - \Delta u = f$ in $R^n \times (0,\infty)$

(iii) $\lim_{(x,t)\to(x^0,0)} u(x,t) = 0, \lim_{(x,t)\to(x^0,0)} u_t(x,t) = 0$ for each point $x^0 \in R^n$ $(x \in R^n, t > 0)$.

Proof: (i) If n is odd, $\left[\frac{n}{2}\right]+1 = \frac{n+1}{2}$ and if n is even , $\left[\frac{n}{2}\right]+1 = \frac{n+2}{2}$

Also $u(.,.;s) \in C^2(R^n \times [s,\infty))$ for each $s \geq 0$ and so $u \in C^2(R^n \times [0,\infty))$.

Hence $u \in C^2(R^n \times [0,\infty))$.

(ii) Differentiating u w.r.t t and x by two times, we have

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds$$

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds$$

Furthermore,

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)\,ds = \int_0^t u_{tt}(x,t;s)\,ds$$

Thus,

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t) \qquad x \in R^n, t \geq 0$$

(iii) And clearly $u(x,0) = u_t(x,0) = 0$ for $x \in R^n$. Therefore equation (2) is the solution of equation (1).

Examples: Let us work out explicitly how to solve (1) for n=1. In this case, d' Alembert's formula gives

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\,dy$$

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dyds$$

i.e. $$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds \qquad (x \in R, t \geq 0) \qquad \ldots (5)$$

For n=3, Kirchhoff's formula implies

$$u(x,t;s) = (t-s) \oint_{\partial B(x,t-s)} f(y,s)\, dS$$

So that

$$u(x,t) = \int_0^t (t-s) \left(\oint_{\partial B(x,t-s)} f(y,s)\, dS \right) ds$$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dS ds$$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS dr$$

Therefore,

$$u(x,t) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\, dy \qquad (x \in R^3, t \geq 0)$$

solves (4) for n=3.

The integrand on the right is called a retarded potential.

4.7 Energy Methods

There is the necessity of making more and more smoothness assumptions upon the data g and h to ensure the existence of a C^2 solution of the Wave equation for large and large n. This suggests that perhaps some other way of measuring the size and smoothness of functions may be more appropriate.

(a) Uniqueness

Let $U \subset R^n$ be a bounded, open set with a smooth boundary ∂U, and as usual set $U_T = U \times (0,T], \Gamma_T = \bar{U}_T - U_T$, where T>0. We are interested in the initial/boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f \ in & U_T \\ \quad u = g \quad on & \Gamma_T \\ \quad u_t = h \quad on & U \times \{t=0\} \end{cases} \qquad \ldots (1)$$

Theorem: There exists at most one function $u \in C^2(\bar{U}_T)$ solving (1).

Proof: If $\tilde{u}$ is another such solution, then $w := u - \tilde{u}$ solves

$$\begin{cases} w_{tt} - \Delta w = 0 \ in & U_T \\ \quad w = 0 \quad on & \Gamma_T \\ \quad w_t = 0 \quad on & U \times \{t=0\} \end{cases}$$

Set “energy”

$$e(t)=\frac{1}{2}\int_U w_t^2(x,t)+|Dw(x,t)|^2\,dx \qquad (0\le t\le T)$$

Differentiating e(t), we have

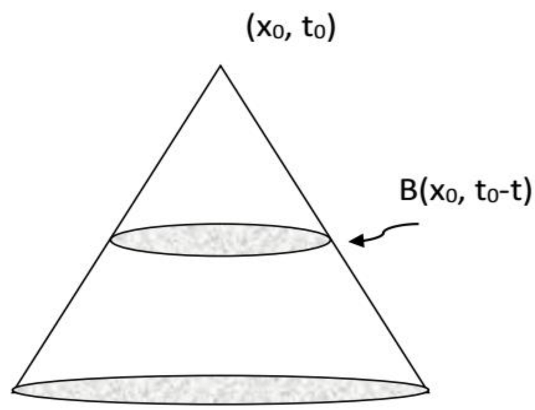
$$\dot{e}(t)=\int_U w_t w_{tt}+Dw.Dw_t dx$$

$$=\int_U w_t(w_{tt}-\Delta w)\,dx=0$$

There is no boundary term since w=0, and hence $w_t=0$, on $\partial U\times[0,T]$. Thus for all $0\le t\le T, e(t)=e(0)=0,$ and so $w_t, Dw=0$ within U_T. Since $w\equiv 0$ on $U\times\{t=0\}$, we conclude $w=u-\tilde{u}=0$ in U_T.

(b) Domain of Dependence

As another illustration of energy methods, let us examine again the domain of dependence of solutions to the Wave equation in all of space.



Cone of dependence

For this, suppose $u\in C^2$ solves

$$u_{tt}-\Delta u=0 \text{ in } R^n\times(0,\infty)$$

Fix $x_0\in R^n, t_0>0$ and consider the cone

$$C=\{(x,t)\,|\,0\le t\le t_0, |x-x_0|\le t_0-t\}.$$